Spanning alternating closed trails in 2-edge-coloured graphs

Jørgen Bang-Jensen

University of Southern Denmark

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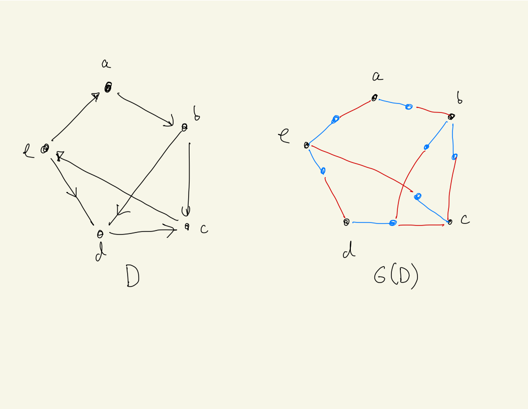
Consider the following conversion of a given digraph $D = (V, A)$ to a 2-edge-coloured bipartite graph $G(D)$: The vertex set of $G(D)$ is $V \cup \{w_{uv} | uv \in A\}$ and the set of edges of $G(D)$ consist of and edge uw_{uv} of colour 1 and an edge $w_{uv}v$ of colour 2 for every arc $uv \in A$.



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Let $G = (V, E)$ be a graph and let $\phi : E \rightarrow \{1, 2\}$ be a 2-edge-colouring of E. A path, cycle,trail or walk X in G is **alternating** if the edges of X alternate between colours 1,2. In figures we represent colour 1 in red and colour 2 in blue.

A graph G is **colour-connected** if there exist two alternating (u, v)-paths P_1, P_2 whose union is an alternating walk for every choice of distinct vertices u, v.

Lemma 1 (Bang-Jensen and Gutin 1998)

One can decide in polynomial time whether a given 2-edge-coloured graph is colour-connected.

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A simpler characterization of colour-connectivity is as follows.

Lemma 2

Let G be a 2-edge-coloured graph. Then G is colour-connected if and only if G has an alternating (u, v)-path starting with colour c for each colour $c \in \{1, 2\}$ and every ordered pair of vertices u, v.

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Let $G = (X, Y, E)$ be a bipartite graph for which each edge is coloured red or blue. Let $D = D(G) = (X, Y, A)$ be the bipartite digraph that we obtain from G by orienting every red edge xy, $x \in X, y \in Y$, as the arc $x \rightarrow y$ and every blue edge $x'y'$,$x' \in X, y' \in Y$, as the arc $y' \rightarrow x'$.

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Proposition 3

The following claims are equivalent for a bipartite digraph D:

(a) *D is strongly connected.*

(b) *$CM(D)$ is colour-connected.*

The following is an immediate consequence of the BB-correspondence and well-known fact that the hamiltonian cycle problem is NP-complete for strongly connected bipartite digraphs.

Theorem 4

It is NP-complete to decide whether a colour-connected 2-edge-coloured bipartite graph has an alternating hamiltonian cycle.

Hamiltonian cycles in 2-edge-coloured complete graphs

The following important theorem due to Bankfalvi and Bankfalvi was originally formulated it in a different, but equivalent way.

Theorem 5 (Bankfalvi and Bankfalvi 1968)

Let H be a 2-edge-coloured complete graph. Then H has an alternating hamiltonian cycle if and only if H is colour-connected and has an alternating cycle factor.

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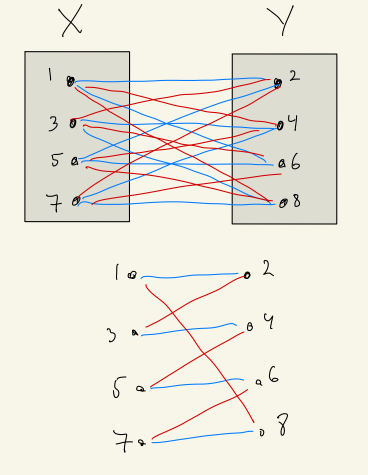
As explained in the next slides this implies the following

Theorem 6

A 2-edge-coloured complete bipartite graph has an alternating hamiltonian cycle if and only if it is colour-connected and has an alternating cycle factor.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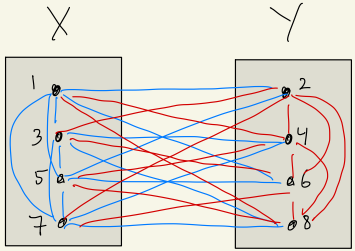

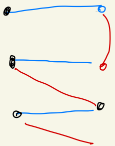

no alternating cycle contains an XX or YY edge

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Theorem 7 (Häggkvist and Manoussakis 1989; Gutin 1984)

A bipartite tournament has a hamiltonian cycle if and only if it is strongly connected and has a cycle factor.

In his PhD thesis, supervised by Manoussakis, Rachid Saad proved the following characterization of the length of a longest alternating cycle in a colour-connected 2-edge-coloured complete graph.

Theorem 8 (Saad 1996)

Let G be a colour-connected 2-edge-coloured complete graph. The length of a longest alternating cycle in G is equal to the maximum number of vertices that can be covered by disjoint alternating cycles in G.

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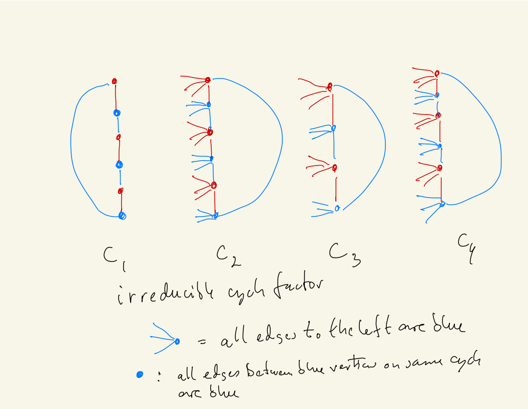
Theorem 8 immediately implies the Bankfalvi and Bankfalvi theorem (Theorem 5).

Irreducible cycle factors



In the figure the 2-edge-coloured complete graph G is not colour-connected since there is no alternating path starting with a blue (red) edge from a red (blue) vertex on C_2 to any vertex of C_1.

For a given alternating cycle factor $C_1, \ldots, C_p$, we write $C_i \rightarrow C_j$ if the relationship between the cycles is as indicated in the figure above where $j > i$.

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Theorem 9 (Bang-Jensen and Gutin 1998)

Let G have an alternating cycle factor $\mathcal{F}$ consisting of $p \geq 2$ cycles. $\mathcal{F}$ is an irreducible alternating cycle factor of G if and only if we can label the cycles in $\mathcal{F}$ as $C_1, \ldots, C_p$, such that, with the notation introduced above, for every $1 \leq i < j \leq p$,
$\chi(X_j V(C_i)) = 1$, $\chi(Y_j V(C_i)) = 2$, $\chi(X_j X_j) = 1$, $\chi(Y_j Y_j) = 2$.

Below we consider a generalization of 2-edge-coloured complete multigraphs, namely those 2-edge-coloured graphs for which the end-vertices of every monochromatic path of length 2 are adjacent, that is, if xyz is a path and $\phi(xy) = \phi(yz)$, then xz is an edge of the graph.

The authors call such graphs **M-closed**.

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Theorem 10 (Contreras-Balbuena, Galeana-Sáanchez and Goldfeder 2019)

Let G be a 2-edge-coloured graph which is M-closed. Then G has an alternating hamiltonian cycle if and only if it is colour-connected and has an alternating cycle factor.

Note the similarity between the condition for being M-closed and the condition for a digraph to be locally semicomplete. A digraph is **locally semicomplete** if the in-neighbourhood and the out-neighbourhood of each vertex induces a semicomplete digraph. The example below shows that this analogy does not extend to in-semicomplete digraphs.

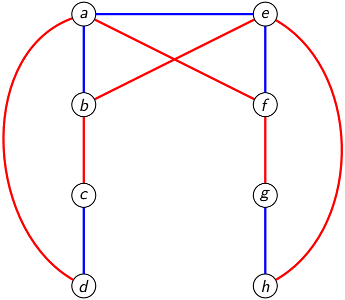

Figure: A 2-edge-coloured graph G in which the end vertices x, z are adjacent for every path xyz with $\phi(xy) = \phi(yz) = 2$ (2=blue). G is colour-connected and has a cycle factor but it has no alternating hamiltonian cycle. It also has no spanning closed alternating trail.

We call a 2-edge-coloured graph G **trail-colour-connected** if G contains two alternating (u, v)-trails T_1, T_2 whose union is an alternating walk for every pair distinct vertices u, v.

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The following analogous of Lemma 2 is easy to derive using almost the same proof as that of Lemma 2.

Lemma 11

Let G be a 2-edge-coloured graph. Then G is trail-colour-connected if and only if G has an alternating (u, v)-trail starting with colour c for each colour $c \in \{1, 2\}$ and every ordered pair of vertices u, v.

Lemma 12 (Bang-Jensen, Bellitto and Yeo 2020)

A 2-edge-coloured complete multipartite graph is colour-connected if and only if it is trail-colour-connected.

Theorem 13 (Bang-Jensen, Bellitto and Yeo 2020)

Let G be a 2-edge coloured graph and let $x, y \in V(G)$ be arbitrary. We can decide if there is a trail from x to y starting with colour c_1 and ending with colour c_2 in polynomial time.

Eulerian factors and supereulerian edge-coloured graphs

- Recall that a connected undirected graph is **eulerian** if it has a spanning closed trail which uses every edge. By Euler's theorem, G is eulerian if and only if it is connected and the degree of every vertex is even. This can be generalized to 2-edge-coloured graphs as follows.

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- Following the standard proof of Euler's theorem is easy to see that a connected 2-edge coloured graph G is eulerian if and only if each vertex v has even degree and half of the edges incident to v have colour i for $i \in [2]$.

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- Following the same definitions for graphs and digraphs, we say that a 2-edge-coloured graph G is **supereulerian** if it contains a spanning closed alternating trail.

An **eulerian factor** of a 2-edge-coloured graph G is a collection of vertex-disjoint induced subgraphs $G_1 = (V_1, E_1), \ldots, G_k = (V_k, E_k)$ of G, such that $V = V_1 \cup \ldots \cup V_k$ and each G_i is supereulerian.

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Let G be a 2-edge-coloured graph. We will construct a new graph, H, such that H has a perfect matching if and only if G has a eulerian factor.

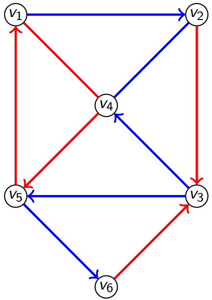

Figure: A 2-edge-coloured graph G with a spanning closed alternating trail in G $v_1 v_2 v_3 v_4 v_5 v_6 v_3 v_5 v_1$ (indicated as directed edges).

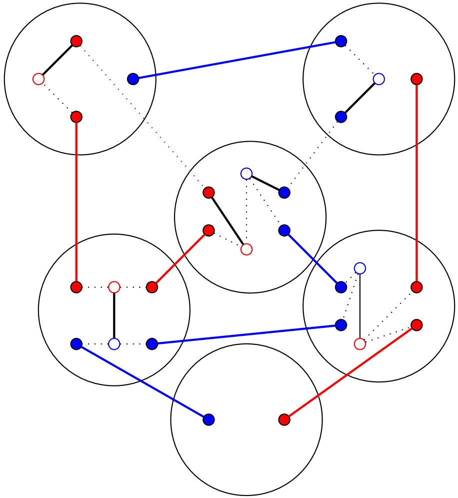

Figure: The graph $H = H(G)$ constructed from the graph G above. The perfect matching corresponding to the spanning eulerian subgraph indicated in the figure above is shown with full lines. The colours are just for easy reference to the other figure.

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Since a bipartite tournament is a semicomplete multipartite digraph, the BB-correspondence implies the following characterization of supereulerian 2-edge-coloured complete bipartite graphs.

Corollary 16

A 2-edge-coloured complete bipartite graph G is supereulerian if and only if G is colour-connected and has an eulerian factor.

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A 2-edge-coloured complete bipartite graph G is supereulerian if and only if G is colour-connected and has an eulerian factor.

As both the problem of deciding if a 2-edge-coloured graph is colour-connected and the problem of deciding if it contains an eulerian factor are polynomial time solvable, we note that Corollary 16 implies that we in polynomial time can decide if a 2-edge-coloured complete bipartite graph is supereulerian.

Let G be a 2-edge-coloured graph on $n > 1$ vertices $\{v_1, v_2, \ldots, v_n\}$. By an **extension** of G we mean any graph $H = G[I_{p_1}, \ldots, I_{p_n}]$ that is obtained from G by replacing each vertex v_i by an independent set $\{v_{i,1}, \ldots, v_{i,p_i}\}$ of $p_i \geq 1$ vertices, $i \in [n]$ and connecting different such sets as follows:

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Extensions of edge-coloured graphs

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Proposition 17

For a 2-edge-coloured graph G the following are equivalent.

(i) *G is colour-connected.*

(ii) *Every extension H of G is colour-connected.*

Theorem 18 (Bang-Jensen, Bellitto and Yeo 2020)

Let G be an extension of an M-closed 2-edge-coloured graph. Then G has an alternating hamiltonian cycle if and only if G is colour-connected and has an alternating cycle factor.

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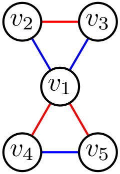

Figure: A non colour-connected graph with a spanning closed alternating trail.

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Theorem 19 (Bang-Jensen, Bellitto and Yeo 2020)

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It is NP-complete to decide if a 2-edge-coloured graph is supereulerian.

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The following is an easy consequence of the fact that A 2-edge-coloured complete multipartite graph is colour-connected if and only if it is trail-colour-connected (by Lemma 12).

Proposition 22

If a 2-edge-coloured complete multipartite graph G has a spanning closed alternating trail, then G is colour-connected.

Proposition 23 (Bang-Jensen, Bellitto and Yeo)

There exists infinitely many 2-edge-coloured complete multipartite graphs which are colour-connected and have an alternating cycle factor but are not super-eulerian.

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Conjecture 24 (Bang-Jensen, Bellitto and Yeo 2020)

There exists a polynomial algorithm for deciding whether a 2-edge-coloured complete multipartite graph is supereulerian.