Spanning alternating closed trails in 2-edge-coloured graphs

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Consider the following conversion of a given digraph $D = (V, A)$ to a 2-edge-coloured bipartite graph $G(D)$: The vertex set of $G(D)$ is $V \cup \{w_{uv} | uv \in A\}$ and the set of edges of $G(D)$ consist of an edge $uw_{uv}$ of colour 1 and an edge $w_{uv}v$ of colour 2 for every arc $uv \in A$. 

![Graph Diagram]
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The converse also holds when the path, trail or walk must start and end in a vertex of $V$. 
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The converse also holds when the path, trail or walk must start and end in a vertex of $V$.

Let $G = (V, E)$ be a graph and let $\phi : E \to \{1, 2\}$ be a 2-edge-colouring of $E$. A path, cycle, trail or walk $X$ in $G$ is **alternating** if the edges of $X$ alternate between colours 1,2. In figures we represent colour 1 in **red** and colour 2 in **blue**.
A graph $G$ is **colour-connected** if there exist two alternating $(u,v)$-paths $P_1, P_2$ whose union is an alternating walk for every choice of distinct vertices $u, v$.

**Lemma 1 (Bang-Jensen and Gutin 1998)**

*One can decide in polynomial time whether a given 2-edge-coloured graph is colour-connected.*
A graph $G$ is **colour-connected** if there exist two alternating $(u, v)$-paths $P_1, P_2$ whose union is an alternating walk for every choice of distinct vertices $u, v$.

**Lemma 1 (Bang-Jensen and Gutin 1998)**

*One can decide in polynomial time whether a given 2-edge-coloured graph is colour-connected.*

A simpler characterization of colour-connectivity is as follows.

**Lemma 2**

*Let $G$ be a 2-edge-coloured graph. Then $G$ is colour-connected if and only if $G$ has an alternating $(u, v)$-path starting with colour $c$ for each colour $c \in \{1, 2\}$ and every ordered pair of vertices $u, v$.***
An **alternating cycle factor** in a 2-edge-coloured graph $G$ is a collection of disjoint alternating cycles that cover $V(G)$. 

Let $G = (X, Y, E)$ be a bipartite graph for which each edge is coloured red or blue. Let $D = D(G) = (X, Y, A)$ be the bipartite digraph that we obtain from $G$ by orienting every red edge $xy$, $x \in X$, $y \in Y$, as the arc $x \to y$ and every blue edge $x' y'$, $x' \in X$, $y' \in Y$, as the arc $y' \to x'$. Now every alternating path, cycle, trail or walk in $G$ corresponds to a directed path, cycle, trail or walk in $D$. 
An **alternating cycle factor** in a 2-edge-coloured graph $G$ is a collection of disjoint alternating cycles that cover $V(G)$.

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Now every alternating path, cycle, trail or walk in $G$ corresponds to a directed path, cycle, trail or walk in $D$. 
It is clear that we can also go the other way by replacing each arc from $X$ to $Y$ by a red edge and each other arc by a blue edge.
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We denote by $CM(D)$ the edge-coloured bipartite graph obtained in the way from $D$. This is called the **BB-correspondence** in BJG Chapter 11.
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**Proposition 3**

The following claims are equivalent for a bipartite digraph $D$:

(a) $D$ is strongly connected.

(b) $CM(D)$ is colour-connected.
The following is an immediate consequence of the BB-correspondence and well-known fact that the hamiltonian cycle problem is NP-complete for strongly connected bipartite digraphs.

**Theorem 4**

*It is NP-complete to decide whether a colour-connected 2-edge-coloured bipartite graph has an alternating hamiltonian cycle.*
The following important theorem due to Bankfalvi and Bankfalvi was originally formulated it in a different, but equivalent way.

**Theorem 5 (Bankfalvi and Bankfalvi 1968)**

Let $H$ be a 2-edge-coloured complete graph. Then $H$ has an alternating hamiltonian cycle if and only if $H$ is colour-connected and has an alternating cycle factor.
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**Theorem 5 (Bankfalvi and Bankfalvi 1968)**

Let $H$ be a 2-edge-coloured complete graph. Then $H$ has an alternating hamiltonian cycle if and only if $H$ is colour-connected and has an alternating cycle factor.

As explained in the next slides this implies the following

**Theorem 6**

A 2-edge-coloured complete bipartite graph has an alternating hamiltonian cycle if and only if it is colour-connected and has an alternating cycle factor.
no alternating cycle contains an XX or YY edge
A bipartite tournament is a bipartite digraph with partition classes $X$ and $Y$ such that there is precisely one arc between each vertex of $X$ and each vertex of $Y$.
A **bipartite tournament** is a bipartite digraph with partition classes $X$ and $Y$ such that there is precisely one arc between each vertex of $X$ and each vertex of $Y$.

A **cycle-factor** in a digraph $D$ is a disjoint collection of cycles $C_1, C_2, \ldots, C_k$ such that $V(D) = V(C_1) \cup \ldots \cup V(C_k)$. 

By using the BB correspondence and Theorem 6, Haggkvist and Manousakis proved the following characterization of Hamiltonian bipartite tournaments.

**Theorem 7** (Haggkvist and Manoussakis 1989; Gutin 1984) A bipartite tournament has a Hamiltonian cycle if and only if it is strongly connected and has a cycle factor.
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**Theorem 7 (Häggkvist and Manoussakis 1989; Gutin 1984)**

*A bipartite tournament has a hamiltonian cycle if and only if it is strongly connected and has a cycle factor.*
In his PhD thesis, supervised by Manoussakis, Rachid Saad proved the following characterization of the length of a longest alternating cycle in a colour-connected 2-edge-coloured complete graph.

**Theorem 8 (Saad 1996)**

Let $G$ be a colour-connected 2-edge-coloured complete graph. The length of a longest alternating cycle in $G$ is equal to the maximum number of vertices that can be covered by disjoint alternating cycles in $G$. 
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Let $G$ be a colour-connected 2-edge-coloured complete graph. The length of a longest alternating cycle in $G$ is equal to the maximum number of vertices that can be covered by disjoint alternating cycles in $G$.

Theorem 8 immediately implies the Bankfalvi and Bankfalvi theorem (Theorem 5).
In the figure the 2-edge-coloured complete graph $G$ is not colour-connected since there is no alternating path starting with a blue (red) edge from a red (blue) vertex on $C_2$ to any vertex of $C_1$. 
For a given alternating cycle factor \( C_1, \ldots, C_p \), we write \( C_i \rightarrow C_j \) if the relationship between the cycles is as indicated in the figure above where \( j > i \).
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Denote by $X_j$ vertices of $C_j$ that send only red edges to the left to the left and by $Y_j$ the vertices of $C_j$ that send only blue edges to the left.
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**Theorem 9 (Bang-Jensen and Gutin 1998)**

Let $G$ have an alternating cycle factor $F$ consisting of $p \geq 2$ cycles. $F$ is an irreducible alternating cycle factor of $G$ if and only if we can label the cycles in $F$ as $C_1, \ldots, C_p$, such that, with the notation introduced above, for every $1 \leq i < j \leq p$,
- $\chi(X_j V(C_i)) = 1$,
- $\chi(Y_j V(C_i)) = 2$,
- $\chi(X_j X_j) = 1$,
- $\chi(Y_j Y_j) = 2$. 

Below we consider a generalization of 2-edge-coloured complete multigraphs, namely those 2-edge-coloured graphs for which the end-vertices of every monochromatic path of length 2 are adjacent, that is, if $xyz$ is a path and $\phi(xy) = \phi(yz)$, then $xz$ is an edge of the graph.

The authors call such graphs **M-closed**.
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**Theorem 10 (Contreras-Balbuena, Galeana-Sáanchez and Goldfeder 2019)**

*Let $G$ be a 2-edge-coloured graph which is M-closed. Then $G$ has an alternating hamiltonian cycle if and only if it is colour-connected and has an alternating cycle factor.*
Note the similarity between the condition for being M-closed and the condition for a digraph to be locally semicomplete. A digraph is **locally semicomplete** if the in-neighbourhood and the out-neighbourhood of each vertex induces a semicomplete digraph. The example below shows that this analogy does not extend to in-semicomplete digraphs.

![Figure: A 2-edge-coloured graph $G$ in which the end vertices $x, z$ are adjacent for every path $xyz$ with $\phi(xy) = \phi(yz) = 2$ (2=blue). $G$ is colour-connected and has a cycle factor but it has no alternating hamiltonian cycle. It also has no spanning closed alternating trail.](image)
We call a 2-edge-coloured graph $G$ **trail-colour-connected** if $G$ contains two alternating $(u, v)$-trails $T_1, T_2$ whose union is an alternating walk for every pair distinct vertices $u, v$. 
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The following analogous of Lemma 2 is easy to derive using almost the same proof as that of Lemma 2.

**Lemma 11**

Let $G$ be a 2-edge-coloured graph. Then $G$ is trail-colour-connected if and only if $G$ has an alternating $(u, v)$-trail starting with colour $c$ for each colour $c \in \{1, 2\}$ and every ordered pair of vertices $u, v$. 
Lemma 12 (Bang-Jensen, Bellitto and Yeo 2020)

A 2-edge-coloured complete multipartite graph is colour-connected if and only if it is trail-colour-connected.

Theorem 13 (Bang-Jensen, Bellitto and Yeo 2020)

Let $G$ be a 2-edge coloured graph and let $x, y \in V(G)$ be arbitrary. We can decide if there is a trail from $x$ to $y$ starting with colour $c_1$ and ending with colour $c_2$ in polynomial time.
Recall that a connected undirected graph is **eulerian** if it has a spanning closed trail which uses every edge. By Euler’s theorem, $G$ is eulerian if and only if it is connected and the degree of every vertex is even. This can be generalized to 2-edge-coloured graphs as follows.
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A 2-edge coloured graph $F$ is **eulerian** if it contains a closed alternating trail which covers all the edges of $G$. 
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Following the standard proof of Euler’s theorem is easy to see that a connected 2-edge coloured graph $G$ is eulerian if and only if each vertex $v$ has even degree and half of the edges incident to $v$ have colour $i$ for $i \in [2]$. 
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Following the same definitions for graphs and digraphs, we say that a 2-edge-coloured graph $G$ is **supereulerian** if it contains a spanning closed alternating trail.
An **eulerian factor** of a 2-edge-coloured graph $G$ is a collection of vertex-disjoint induced subgraphs $G_1 = (V_1, E_1), \ldots, G_k = (V_k, E_k)$ of $G$, such that $V = V_1 \cup \ldots \cup V_k$ and each $G_i$ is supereulerian.
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There exists a polynomial algorithm for finding an eulerian factor of a 2-edge-coloured graph $G$ or producing a certificate that $G$ has no such factor.
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**Lemma 14**

*There exists a polynomial algorithm for finding an eulerian factor of a 2-edge-coloured graph $G$ or producing a certificate that $G$ has no such factor.*

Let $G$ be a 2-edge-coloured graph. We will construct a new graph, $H$, such that $H$ has a perfect matching if and only if $G$ has a eulerian factor.
Figure: A 2-edge-coloured graph $G$ with a spanning closed alternating trail in $G$ $v_1 v_2 v_3 v_4 v_5 v_3 v_5 v_1$ (indicated as directed edges).
Figure: The graph $H = H(G)$ constructed from the graph $G$ above. The perfect matching corresponding to the spanning eulerian subgraph indicated in the figure above is shown with full lines. The colours are just for easy reference to the other figure.
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**Theorem 15 (Bang-Jensen and Maddaloni 2015)**

A semicomplete multipartite digraph is supereulerian if and only if it is is strongly connected and has an eulerian factor.

Since a bipartite tournament is a semicomplete multipartite digraph, the BB-correspondence implies the following characterization of supereulerian 2-edge-coloured complete bipartite graphs.

**Corollary 16**

A 2-edge-coloured complete bipartite graph $G$ is supereulerian if and only if $G$ is colour-connected and has an eulerian factor.
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**Corollary 16**

A 2-edge-coloured complete bipartite graph $G$ is supereulerian if and only if $G$ is colour-connected and has an eulerian factor.

As both the problem of deciding if a 2-edge-coloured graph is colour-connected and the problem of deciding if it contains an eulerian factor are polynomial time solvable, we note that Corollary 16 implies that we in polynomial time can decide if a 2-edge-coloured complete bipartite graph is supereulerian.
Let $G$ be a 2-edge-coloured graph on $n > 1$ vertices $\{v_1, v_2, \ldots, v_n\}$. By an extension of $G$ we mean any graph $H = G[I_{p_1}, \ldots, I_{p_n}]$ that is obtained from $G$ by replacing each vertex $v_i$ by an independent set $\{v_{i,1}, \ldots, v_{i,p_i}\}$ of $p_i \geq 1$ vertices, $i \in [n]$ and connecting different such sets as follows:
Let $G$ be a 2-edge-coloured graph on $n > 1$ vertices \{v_1, v_2, \ldots, v_n\}. By an extension of $G$ we mean any graph $H = G[\{l_{p_1}, \ldots, l_{p_n}\}]$ that is obtained from $G$ by replacing each vertex $v_i$ by an independent set \{v_{i,1}, \ldots, v_{i,p_i}\} of $p_i \geq 1$ vertices, $i \in [n]$ and connecting different such sets as follows:

If $v_i v_j$ is an edge in $G$ of colour $c$ then $H$ contains an edge of colour $c$ between $v_{i,q}$ and $v_{j,r}$ for every choice of $q \in [p_i], r \in [p_j]$. 

Proposition 17

For a 2-edge-coloured graph $G$ the following are equivalent.

(i) $G$ is colour-connected.

(ii) Every extension $H$ of $G$ is colour-connected.
Extensions of edge-coloured graphs

Let $G$ be a 2-edge-coloured graph on $n > 1$ vertices \{\(v_1, v_2, \ldots, v_n\}\). By an extension of $G$ we mean any graph $H = G[I_{p_1}, \ldots, I_{p_n}]$ that is obtained from $G$ by replacing each vertex $v_i$ by an independent set \(\{v_{i,1}, \ldots, v_{i,p_i}\}\) of $p_i \geq 1$ vertices, $i \in [n]$ and connecting different such sets as follows:
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Theorem 18 (Bang-Jensen, Bellitto and Yeo 2020)

Let $G$ be an extension of an M-closed 2-edge-coloured graph. Then $G$ has an alternating hamiltonian cycle if and only if $G$ is colour-connected and has an alternating cycle factor.
Theorem 18 (Bang-Jensen, Bellitto and Yeo 2020)

Let $G$ be an extension of an $M$-closed 2-edge-coloured graph. Then $G$ has an alternating hamiltonian cycle if and only if $G$ is colour-connected and has an alternating cycle factor.

Armed with Theorem 18 we are now ready to characterize supereulerian extensions of $M$-closed 2-edge-coloured graphs.
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Note that, by the example below, a supereulerian $M$-closed 2-edge-coloured graph does not have to be colour-connected, but it must be trail-colour-connected.
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Note that, by the example below, a supereulerian M-closed 2-edge-coloured graph does not have to be colour-connected, but it must be trail-colour-connected.

![A non colour-connected graph with a spanning closed alternating trail.](image)
The figure shows that for M-closed 2-edge-coloured graphs, having a spanning closed alternating trail does not imply colour-connectivity.

Note that the graph is trail-colour-connected, as is every 2-edge-coloured graph with a spanning closed alternating trail.
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**Theorem 19 (Bang-Jensen, Bellitto and Yeo 2020)**

Let $G$ be an extension of an M-closed 2-edge-coloured graph. Then $G$ is supereulerian if and only if it is trail-colour-connected and has an eulerian factor.
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**Theorem 19 (Bang-Jensen, Bellitto and Yeo 2020)**

*Let $G$ be an extension of an M-closed 2-edge-coloured graph. Then $G$ is supereulerian if and only if it is trail-colour-connected and has an eulerian factor.*

**Theorem 20**

*It is NP-complete to decide if a 2-edge-coloured graph is supereulerian.*
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**Problem 21**

What is the complexity of deciding whether a 2-edge-coloured complete multipartite graph has an alternating Hamiltonian cycle? Is there a good characterization?
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**Problem 21**

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The following is an easy consequence of the fact that a 2-edge-coloured complete multipartite graph is colour-connected if and only if it is trail-colour-connected (by Lemma 12).

**Proposition 22**

*If a 2-edge-coloured complete multipartite graph $G$ has a spanning closed alternating trail, then $G$ is colour-connected.*
Proposition 23 (Bang-Jensen, Bellitto and Yeo)

There exists infinitely many 2-edge-coloured complete multipartite graphs which are colour-connected and have an alternating cycle factor but are not super-eulerian.
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Conjecture 24 (Bang-Jensen, Bellitto and Yeo 2020)

There exists a polynomial algorithm for deciding whether a 2-edge-coloured complete multipartite graph is supereulerian.